This paper is available online at www.praiseworthyprize.org
International Review of Automatic Control (I.RE.A.CO.), Vol. 12, N. 4
ISSN 1974-6059 July 2019

Improved Mode-Dependent State-Feedback Stabilization of Discrete-Time Networked Control Systems with Markovian Communication Delays

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Abstract – *The modeling issues in dynamical systems in many processes, networked control systems (NCS) are very complex. The models may contain subsystems with different parameters, which arise when using a network in an NCS such as time delay, limited bandwidth, and so on. Controlling these types of multiple time delay system is challenging, due to mathematical complexity. This paper considers the design of a stabilizing state-feedback controller for a networked control system with random communication delays. Sensor-to-controller (S-C) and controller-to-actuator delays are modeled by two independent Markov chains. Network-induced random delays are modeled as a Markov chain, and the resulting closed-loop system is transformed into a Markovian jump linear system (MJLS). The focus is on the design of a controller that fully incorporates the effect of the C-A delay. The resulting closed-loop system is described by a new discrete-time Markovian jump linear system with Markov delays model. Then, by applying a type of stochastic Lyapunov functional, sufficient conditions on the stochastic stabilizability and the existence of controller are derived in terms of coupled linear matrix inequalities (LMIs). The efficacy of the proposed method is shown through illustrative examples. Simulation results demonstrate the applicability and the effectiveness of the obtained theoretical results.* ***Copyright © 2019 The Authors.***

***Keywords**: Discrete-Time Markovian Jump Linear Systems, Markov Chains, Networked Control Systems, Random Delays, Linear Matrix Inequality, Stochastic Stability*

Nomenclature

T	Matrix transposition
$\mathbb{R}^n$	n-Dimensional Euclidean space
$\mathbb{E}[\cdot]$	Mathematical expectation
K, P, Q_v, Z_v	Matrices
$\mathcal{U}_d, \mathcal{V}_d, \mathcal{W}_d, \Psi_{1,2,3,4}$	Matrices
$[\mathrm{T}^{ca}], [\mathrm{T}^{ca}]$	Transition probability matrices
$\mathcal{M}, \mathcal{N}$	Markov chains time sets
τ_k^{sc}	Time delay sensor to controller
τ_k^{ca}	Time delay controller to actuator
$I, 0$	Identity matrix and zero matrix
$\boldsymbol{K}$	Control signal
λ_{min}, ρ_{ij}	Minimal Eigenvalues
$\bar{u}$	Control signal
*	Ellipsis for terms induced by symmetry
$diag\{\dots\}$	Diagonal matrix

I. Introduction

In Networked Control Systems (NCS's), sensors, actuators and controllers are spatially distributed over wide areas with data exchanges occurring through a shared band-limited digital transmission channel according to standardized communication protocols such as the User Datagram Protocol (UDP) and the Transfer Control Protocol (TCP). Compared with the traditional centralized point-to-point control systems, the use of networking technologies for remote data transfers among spatially distributed devices of a control application provides several attractive benefits such as functional modularity, flexibility in system design architectures, reduced wiring complexity, lower cabling costs, improved reliability, and ease in installation and maintenance [1]-[5]. As a result of these important advantages, NCSs have become ubiquitous with applications ranging from smart automobiles to factory automation, industrial process control and large electric power networks. However, despite their many attractive features, the use of NCSs has introduced new challenges and constraints of significant interest to the control research community. Indeed, the insertion of an 'imperfect' communication channel between the plant and controller can strongly degrade the performance of an otherwise well-behaving control system, and induces undesirable stability problems. The causes of poor performance in NCSs are not only inherent to the physical nature of the network, but also to the conditions of its operation. Specifically, the most challenging issues in NCSs research are network-induced time delays,

Available online by December 31st, 2020 https://doi.org/10.15866/ireaco.v12i4.16342

information loss and packet disorder [6]-[9]. While data packet losses and disorder phenomenon occur because of network traffic congestion and ‘failures’ in the delivery of data packets from source to destination through the network, the causes of network-induced delays, in both sensor-to-controller (S-C or forward) and controller-to-actuator (C-A or backward) links, include the processing time required to encapsulate measurement or control data into packets, the queuing time for network availability, in addition to the propagation time taken by a data packet to travel from source to destination through the network’s physical media [1], [2]. The effect of network-induced delays on NCSs has attracted considerable attention, and many approaches have been developed in order to deal with the problem. Depending on the type of network and on the used communication protocols, existing results can be categorized into three main pairs [7]: 1) constant versus time-varying delay; 2) deterministic versus stochastic delays; and 3) delay smaller than one sampling interval and otherwise. For stochastically varying delays with the realistic assumption of some correlation between the current delay and the previous ones, the variation may be associated with some statistical descriptions and can be effectively modeled using the Markov chains.

This allows the stability analysis and the controller design of such NCSs to be cast within the framework of Discrete-time Markovian jump linear systems (DMJLS) with delay [10]-[15], [33], [34]. The problem of NCSs under random communication delay constraints have been extensively investigated over the last two decades. In Krtolica et al. [16], time-varying delays have been modeled by Markov chains and, complete conditions for zero-state mean-square exponential closed-loop stability are established for linear discrete-time systems using a stochastic Lyapunov function approach. In Nilsson et al. [17], an optimal stochastic state-feedback control structure has been proposed where the control and measurement signals are supplemented time-stamped. The control law has been derived using the same technique as for the standard linear quadratic gaussian (LQG) problem.

Another setup with time-stamped data in the network communication has been studied in Xiao et al. [18] where the mode-dependent state feedback control problem has been converted into an output feedback control problem and solved within the framework of Markovian jump linear systems (MJLS) as a convex optimization over a set of LMI's. In [19], H. Lin et al. have considered both time-delay and packet dropout issues of NCSs in the framework of discrete-time switched systems. Sufficient conditions for stability under state feedback control law and H_∞ disturbance attenuation have been studied. Further interesting results on stabilization of discrete-time NCSs with random delays have been obtained by L. Zhang et al. [20] and Y. Shi et al. [21]. In [20], a two-mode state feedback controller whose gain depends on both current sensor-to-controller delay (τ_k^{sc}) and previous controller-to-actuator delay (τ_{k-1}^{ca}) has been calculated using an iterative linear matrix inequality (LMI) approach. In [21], multi-step Markovian delay mode jumps have been involved in the design of a stabilizing two-mode dynamic output feedback controller by solving a set of LMIs. In the work of J. Wu et al. [22], a NCS model for random packet dropouts has been proposed where both controller and the actuator have been fed through buffers with last-in-first-out (LIFO) rule. In this setup, consecutive packet losses have been treated as random varying delays and, a mode-dependent state feedback controller is calculated by solving linear matrix inequalities. In more recent studies, several advanced control schemes and setups for discrete-time NCSs with random delays have been proposed in order to ensure better performance and stability. In Q. Li et al. [23], the mixed H_2/H_∞ control problem has been investigated and sufficient conditions have been established regarding the existence of a two-mode state feedback H_2/H_∞ controller with guaranteed closed-loop stochastic stability. In [24], L. Qui et al. have presented a unified model to cast random delay and packet dropouts simultaneously. In [25], M.S. Mahmoud et al. have extended the work in [23] to focus on discrete-time networked control systems with plant uncertainty and random Markovian delays with partially known transition probabilities. The methods developed in [20], [21] have been improved in [27] by considering uncertainties in the transition matrices due to time-variations or incomplete statistic information. Other results involving NCSs with random delays can be found in [28]-[31]. So far, the design of mode-dependent state feedback controllers to stabilize NCSs with delays in both sensor-to-controller (τ_k^{sc}) and controller-to-actuator (τ_k^{ca}) links has been carried out with the assumption that the total delay imposed on the state signal through the control loop (sensor-controller-actuator) is $\tau_k^{ca}+\tau_k^{sc}$. However, this is only an approximation that does not account for the effect of τ_k^{ca} on τ_k^{sc} at the controller-actuator link. Indeed, if at a given sample time k the state information $x(k)$ is time-shifted by $\tau_k^{sc}(k \leftarrow k-\tau_k^{sc})$ to be available to the controller as $x(k-\tau_k^{sc})$ due to the S-C link latency, the same is true when this delayed state travels through the C-A link where the time is again shifted by $\tau_k^{ca}(k \leftarrow k-\tau_k^{ca})$ so that the state reaches the actuator as $x(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc})$ instead of $x(k-\tau_k^{ca}\tau_k^{sc})$ assumed in the available literature. To the best of the authors’ knowledge, this approximation has only been mentioned and justified in [24] but the design of a mode-dependent state feedback controller for non-approximated problem accounting for the effect of the C-A delay has not been investigated, which motivates the focus of this paper. In this work, the design of a stabilizing state-feedback controller for a networked control system with C-A and C-S Markovian delays within a DMJLS framework has been considered. It is assumed that at each sampling instant , the current S-C delay τ_k^{sc} and the previous C-A delay τ_{k-1}^{ca} are available to the controller and actuator nodes, respectively, by the time-stamping technique. However, the C-A delay τ_{k-1}^{ca} travels through the S-C link to reach the controller as $\tau^{ca}_{k-\tau^{sc}_k-1}$. Consequently, the two-mode controller gain can be designed to depend on τ^{sc}_k and $\tau^{ca}_{k-\tau^{sc}_k-1}$. More importantly, the full effect of the C-A delay τ^{ca}_k on the state signal is incorporated in the closed-loop model in order to reduce further the conservativeness of the stabilization conditions of the NCS. In addition, unlike most of the aforementioned references, the proposed design in this paper avoids the use of the state variable augmentation approach, which leads to a significant reduction in computational complexity of the scheme.

This paper is organized as follows. In Section II, the control problem formulation of the NCS including the Markovian delay description is presented. Section III establishes the mode-dependent stability conditions with the state-feedback controller. Simulation examples are given in order to illustrate the proposed design in Section IV. Conclusions are summarized in Section V.

II. Control Problem Formulation

The networked control system illustrated in Fig. 1 is considered, where the plant is a discrete linear time-invariant system described by the following state-space equation:

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $x(k) \in \mathbb{R}^p$ is the plant state vector, $u(k) \in \mathbb{R}^q$ is the plant control input. A and B are known real-valued constant system matrices with appropriate dimensions.

The plant, sensor, actuator and controller are spatially distributed with data exchanges occurring through a communication network. Random time delays exist in the S-C and C-A links and, are denoted by τ^{sc}_k and τ^{ca}_k, respectively. The delays are bounded scalars, i.e. $0 \le \tau^{sc}_{min} \le \tau^{sc}_k \le \tau^{sc}_{max}$, $0 \le \tau^{ca}_{min} \le \tau^{ca}_k \le \tau^{ca}_{max}$. A reasonable way to model the delays τ^{sc}_k and τ^{ca}_k is to use finite state homogeneous Markov chains to take into account the correlations between the current delays and the previous delays [12], [13], [19], [21], [22], [24], [28].

In this paper, τ^{sc}_k and τ^{ca}_k are modeled as two independent Markov chains that take values in the finite sets $\mathcal{M} = \{0,1, \ldots, \tau^{sc}_{max}\}$ and $\mathcal{N} = \{0,1, \ldots, \tau^{ca}_{max}\}$.

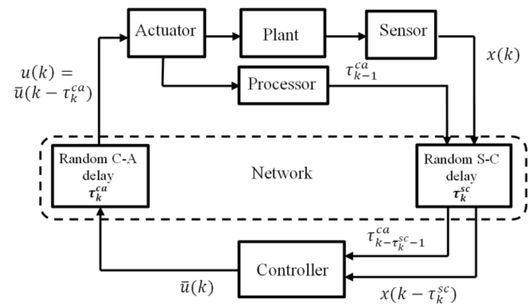

Fig. 1. Structure of Networked control system

Their transition probability matrices are $\mathrm{T}^{sc} = \{\lambda_{mn}\}$ and $\mathrm{T}^{ca} = \{\rho_{ij}\}$, respectively. The transition probabilities of τ^{sc}_k and τ^{ca}_k jumping from mode m to n and from mode i to j, respectively, are defined by:

$$\begin{aligned} \lambda_{mn} &= Pr(\tau^{sc}_{k+1} = n | \tau^{sc}_k = m) \\ \rho_{ij} &= Pr(\tau^{ca}_{k+1} = j | \tau^{ca}_k = i) \end{aligned} \tag{2}$$

where $\lambda_{mn} \ge 0$ and $\rho_{ij} \ge 0$, and:

$$\sum_{n=0}^{\tau^{sc}_{max}} \lambda_{mn} = 1, \quad \sum_{j=0}^{\tau^{ca}_{max}} \rho_{ij} = 1 \tag{3}$$

for all $m, n \in \mathcal{M}$ and $i, j \in \mathcal{N}$. At the actuator node where the control signal is supplemented time-stamped, an embedded processor calculates the delay information τ^{ca}_{k-1} at every sample time k, before its transmission to the controller. At the controller node, while the delay τ^{sc}_k is obtained by the time stamping technique, the information τ^{ca}_{k-1} is not immediately accessible due to the network induced delay at the S-C link. Consequently, the information available to the controller, at every sample time k, is the current S-C delay τ^{sc}_k an the old C-A delay $\tau^{ca}_{k-\tau^{sc}_k-1}$. Therefore, the state feedback control action $\bar{u}(k)$ can be computed by the remote controller based on the delayed state $x(k-\tau^{sc}_k)$ and the latest delay information available at time k, namely, τ^{sc}_k and $\tau^{ca}_{k-\tau^{sc}_k-1}$:

$$\bar{u}(k) = \boldsymbol{K}\left(\tau^{sc}_k, \tau^{ca}_{k-\tau^{sc}_k-1}\right) x(k-\tau^{sc}_k) \tag{4}$$

where $\boldsymbol{K}\left(\tau^{sc}_k, \tau^{ca}_{k-\tau^{sc}_k-1}\right)$ is the two-mode delay-dependent state feedback controller gain. When the control signal $\bar{u}(k)$ is transmitted from the controller, it is further delayed by τ^{ca}_k before it reaches the actuator as:

$$u(k) = \bar{u}(k-\tau^{ca}_k) \tag{5}$$

with:

$$\bar{u}(k-\tau^{ca}_k) = \boldsymbol{K}\left(\tau^{sc}_k, \tau^{ca}_{k-\tau^{sc}_k-1}\right) x\left(k-\tau^{ca}_k-\tau^{sc}_{k-\tau^{ca}_k}\right) \tag{6}$$

Using (5) and (6), Equation (1) can be rewritten in delayed closed loop system as:

$$\begin{aligned} x(k+1) = {} & Ax(k) + B\boldsymbol{K}\left(\tau^{sc}_k, \tau^{ca}_{k-\tau^{sc}_k-1}\right) \\ & \cdot x(k-\tau^{ca}_k-\tau^{sc}_{k-\tau^{ca}_k}) \end{aligned} \tag{7}$$

with initial condition:

$$x(i) = \varphi_i, \ i = -\tau^{ca}_{max} - \tau^{sc}_{max}, -\tau^{ca}_{max} - \tau^{sc}_{max} + 1, \ \ldots, \ 0.$$

Remark 1: Equation (6) is obtained from (4) by carrying out a time shift of τ^{ca}_k, thus replacing time k by $(k-\tau^{ca}_k)$ in the control signal $\bar{u}(k)$. However, because the modes of the gain $\boldsymbol{K}\left(\tau^{sc}_k, \tau^{ca}_{k-\tau^{sc}_k-1}\right)$ are computed and fixed at the controller node, they are not affected by the induced C-A link delay τ_k^{ca}.

Remark 2: Although τ_k^{sc} and τ_k^{ca} are random delays driven by discrete-time Markov chains, the closed-loop system (6) is not a standard two-mode delayed Markovian jump linear system (MJLS), because the state feedback controller gain depends on the modes τ_k^{sc} and $\tau^{ca}_{k-\tau_k^{sc}-1}$ which are related, simultaneously, both to τ_k^{pc} and τ_k^{cp}.

Remark 3: For $\tau_k^{sc} = m$ and $\tau^{ca}_{k-\tau_k^{sc}-1} = i$, the closed-loop system (7) is said to be in mode (m, i) and the controller gain $\boldsymbol{K}\left(\tau_k^{sc}, \tau^{ca}_{k-\tau_k^{sc}-1}\right)$ is denoted by $\boldsymbol{K}(m, i)$.

The objective is to design a state feedback controller such that the closed-loop system (7) is stochastically stable. In this setup, the full effect of the C-A delay τ_k^{ca} on the state signal is incorporated in the closed-loop model in order to reduce further the conservativeness of the stabilization conditions of the NCS. In addition, unlike most of the available references, the proposed state feedback design in this paper avoids the use of the state variable augmentation approach which leads to significant reduction in computational complexity of the scheme. For the underlying system, the following definition is adopted for stochastic stability [23].

Definition 1: The closed-loop system in (7) is said to be stochastically stable if for every initial condition $\varphi_i \in \mathbb{R}^p$ defined on $\mathcal{I} = \{-\tau_{max}^{ca} - \tau_{max}^{sc}, -\tau_{max}^{sc} + 1, \ . \ . \ . \ , 0\}$ and initial delay modes $\tau_0^{ca}, \tau_0^{sc}, \tau^{sc}_{-\tau_0^{ca}}$, $\tau^{ca}_{-\tau_0^{cp}-1}$. A constant ξ exists, such that:

$$\sum_{k=0}^{\infty} \mathbb{E}\left[\|x(k)\|^2\, x(0), \tau_0^{ca}, \tau_0^{sc}, \tau^{sc}_{-\tau_0^{ca}}, \tau^{ca}_{-\tau_0^{sc}-1}\right] \leq \xi \Upsilon \begin{pmatrix} \varphi_i, -\tau_{max}^{sc} - \tau_{max}^{ca} \leq i \leq \\ 0, \tau_0^{ca}, \tau_0^{sc}, \tau^{sc}_{-\tau_0^{ca}}, \tau^{ca}_{-\tau_0^{sc}-1} \end{pmatrix} \tag{8}$$

where:

$$\Upsilon(\varphi_i, -\tau_{max}^{sc} - \tau_{max}^{ca} \leq i \leq 0, \tau_0^{ca}, \tau_0^{sc}, \tau^{sc}_{-\tau_0^{ca}}, \tau^{ca}_{-\tau_0^{sc}-1})$$

is a nonnegative function of the system initial values satisfying $\Upsilon(0,0,\ldots,0) = 0$.

III. Main Results

In this section, sufficient conditions for the stochastic stability of the closed loop system (7) are established and a two-mode-dependent state feedback stabilizing controller is designed by applying a stochastic Lyapunov functional and linear matrix inequality approach. As the controller gain depends on mode delay $\tau^{ca}_{k-\tau_k^{sc}-1}$, the multi-step jump of Markov chains is involved in system (7). Therefore, the transition probability matrix for the multi-step delay mode jump is applicable in designing the state feedback controller. For this purpose, the following proposition is given.

Proposition 1 [21]: If the transition probability matrix from τ_{k-1}^{ca} to τ_k^{ca} is $\mathrm{T}^{ca} = \{\rho_{ij}\}$, then the transition probability matrix from $\tau^{ca}_{k-\tau^{sc}_{k+1}}$ to τ_k^{ca} is $[\mathrm{T}^{ca}]^{\tau^{sc}_{k+1}}$, which is still a transition probability matrix of the Markov chain. Especially when $\tau^{sc}_{k+1} = 0$, the transition probability matrix is $[\mathrm{T}^{ca}]^{\tau^{sc}_{k+1}} = [\mathrm{T}^{ca}]^0 = I$, where:

$$[\mathrm{T}^{ca}]^{\tau^{sc}_{k+1}} = \sum_{j_{\tau^{sc}_{k+1}}}^{\tau^{ca}_{max}} \cdots \sum_{j_3=0}^{\tau^{ca}_{max}} \sum_{j_2=0}^{\tau^{ca}_{max}} \begin{bmatrix} \rho_{0j_2}\rho_{j_2j_3}\cdots\rho_{j_{\tau^{sc}_{k+1}}0} & \rho_{0j_2}\rho_{j_2j_3}\cdots\rho_{j_{\tau^{sc}_{k+1}}1} & \cdots & \rho_{0j_2}\rho_{j_2j_3}\cdots\rho_{j_{\tau^{sc}_{k+1}}\tau^{ca}_{max}} \\ \rho_{1j_2}\rho_{j_2j_3}\cdots\rho_{j_{\tau^{sc}_{k+1}}0} & \rho_{1j_2}\rho_{j_2j_3}\cdots\rho_{j_{\tau^{sc}_{k+1}}1} & \cdots & \rho_{1j_2}\rho_{j_2j_3}\cdots\rho_{j_{\tau^{sc}_{k+1}}\tau^{ca}_{max}} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{\tau^{ca}_{max}j_2}\rho_{j_2j_3}\cdots\rho_{j_{\tau^{sc}_{k+1}}0} & \rho_{\tau^{ca}_{max}j_2}\rho_{j_2j_3}\cdots\rho_{j_{\tau^{sc}_{k+1}}1} & \cdots & \rho_{\tau^{ca}_{max}j_2}\rho_{j_2j_3}\cdots\rho_{j_{\tau^{sc}_{k+1}}\tau^{ca}_{max}} \end{bmatrix} \tag{9}$$

The sufficient conditions to guarantee stochastic stability of system (7) according to definition 1 are shown in the following theorem.

Theorem 1: For the closed-loop system (7) with random but bounded network induced delays $\tau_k^{sc} \in \mathcal{M}$ and $\tau_k^{ca} \in \mathcal{N}$, if for each mode $m \in \mathcal{M} = \{0,1, \ldots, \tau_{max}^{sc}\}$ and $i \in \mathcal{N} = \{0,1, \ldots, \tau_{max}^{ca}\}$, there exist matrices $Q_v > 0$, $Z_v > 0$, $v = 1,2$, $\mathcal{U}_d(m,i)$, $\mathcal{V}_d(m,i)$, $\mathcal{W}_d(m,i)$, $d = 1,2,3$ $P(m,i) > 0$ and $K(m,i)$, satisfying the following matrix inequality:

$$\begin{bmatrix} -\bar{P}(m,i) & 0 & 0 & \Psi_1(m,i) \\ * & -Z_2 & 0 & \Psi_2(m,i) \\ * & * & -Z_1 & \Psi_3(m,i) \\ * & * & * & \Psi_4(m,i) \end{bmatrix} < 0 \tag{10}$$

where:

$$\Psi_1(m,i) \triangleq [\bar{P}(m,i)A \quad \bar{P}(m,i)BK(m,i) \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\Psi_2(m,i) \triangleq [\sqrt{\tau_M}Z_2(A-I) \quad \sqrt{\tau_M}Z_2BK(m,i) \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\Psi_3(m,i) \triangleq [\sqrt{\tau_M}Z_1(A-I) \quad \sqrt{\tau_M}Z_1BK(m,i) \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\Psi_4(m,i) \triangleq \begin{bmatrix} \Lambda_{11}(m,i) & \Lambda_{12}(m,i) & \Lambda_{13}(m,i) & \sqrt{\tau_M}\mathcal{U}_1(m,i) & \sqrt{\tau_r}\mathcal{V}_1(m,i) & \sqrt{\tau_M}\mathcal{W}_1(m,i) \\ * & \Lambda_{22}(m,i) & \Lambda_{23}(m,i) & \sqrt{\tau_M}\mathcal{U}_2(m,i) & \sqrt{\tau_r}\mathcal{V}_2(m,i) & \sqrt{\tau_M}\mathcal{W}_2(m,i) \\ * & * & \Lambda_{33}(m,i) & \sqrt{\tau_M}\mathcal{U}_3(m,i) & \sqrt{\tau_r}\mathcal{V}_3(m,i) & \sqrt{\tau_M}\mathcal{W}_3(m,i) \\ * & * & * & -Z_1 & 0 & 0 \\ * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & -Z_2 \end{bmatrix}$$

with:

$$\bar{P}(m,i) \triangleq \sum_{n=0}^{\tau_{max}^{sc}} \sum_{j=0}^{\tau_{max}^{ca}} \sum_{s_1=0}^{\tau_{max}^{ca}} \lambda_{mn} [T^{ca}]_{i,j}^{m-n+1} [T^{ca}]_{j,s_1}^{n} P(n,j)$$

and:

$$\Lambda_{11}(m,i) \triangleq \mathcal{U}_1(m,i) + \mathcal{U}_1^T(m,i) + \mathcal{W}_1(m,i) + \mathcal{W}_1^T(m,i) - P(m,i) + (1+\tau_M)Q_1 + Q_2$$

$$\Lambda_{12}(m,i) \triangleq -\mathcal{U}_1(m,i) + \mathcal{U}_2^T(m,i) - \mathcal{V}_1 + \mathcal{W}_2^T(m,i)$$

$$\Lambda_{13}(m,i) \triangleq \mathcal{U}_3^T(m,i) - \mathcal{V}_1(m,i) - \mathcal{W}_1(m,i) + \mathcal{W}_3^T(m,i)$$

$$\Lambda_{22}(m,i) \triangleq -\mathcal{U}_2(m,i) - \mathcal{U}_2^T(m,i) + \mathcal{V}_2(m,i) + \mathcal{V}_2^T(m,i) - P(m,i) - Q_1$$

$$\Lambda_{23}(m,i) \triangleq -\mathcal{U}_3^T(m,i) - \mathcal{V}_2(m,i) - \mathcal{V}_3^T(m,i) - \mathcal{W}_2(m,i)$$

$$\Lambda_{33}(m,i) \triangleq -\mathcal{V}_3(m,i) - \mathcal{V}_3^T(m,i) - \mathcal{W}_3(m,i) - \mathcal{W}_3^T(m,i) - Q_2$$

$$\tau_M = \tau_{max}^{ca} + \tau_{max}^{sc}, \tau_m = \tau_{min}^{ca} + \tau_{min}^{sc}, \tau_r = \tau_M - \tau_m$$

Then, the closed-loop system (7) driven by controller (4) is stochastically stable.

Proof: For the closed-loop system (7), the following stochastic Lyapunov functional candidate is constructed:

$$V\left(x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right) = \sum_{s=1}^{5} V_s\left(x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right)$$

where:

$$V_1\left(x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right) = x^T(k) P\left(\tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}\right) x(k) = x^T(k) P(m,i) x(k)$$

$$V_2\left(x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right) = \sum_{l=k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}}^{k-1} x^T(l) Q_1 x(l)$$

$$V_3\left(x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right) = \sum_{\theta=-\tau_{max}^{ca}-\tau_{max}^{sc}+1}^{-\tau_{min}^{ca}-\tau_{min}^{sc}} \sum_{l=k+\theta}^{k-1} x^T(l) Q_1 x(l)$$

$$V_4\left(x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right) = \sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}}^{k-1} x^T(l) Q_2 x(l)$$

$$V_5\left(x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right) = \sum_{\theta=-\tau_{max}^{ca}-\tau_{max}^{sc}}^{-1} \sum_{l=k+\theta}^{k-1} \delta^T(l)(Z_1+Z_2)\delta(l)$$

where $\delta(l) = x(l+1) - x(l)$ and $P(m,i)$, Q_1, Q_2, Z_1 and Z_2 satisfying (10). In order to simplify the equations, the subscripts in the parentheses $(x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc})$, will be omitted and replaced by dot $(\cdot)$, in $V_1 \ldots V_5$. The following is denoted:

$$\Delta V = \sum_{s=1}^{5} \Delta V_s = \sum_{s=1}^{5} \mathbb{E} \left[V_s\left(x(k+1), \tau_{k+1}^{sc}, \tau_{k-\tau_{k+1}^{sc}}^{ca}, \tau_{k+1}^{ca}, \tau_{k+1-\tau_{k+1}^{ca}}^{sc} \mid x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right) - V_s(\cdot)\right. \tag{11}$$

Define $\tau_k^{sc} = m\tau_{k+1}^{sc} = n$, $\tau_{k-\tau_k^{sc}-1}^{ca} = i$, $\tau_{k-\tau_{k+1}^{sc}}^{ca} = j$, $\tau_k^{ca} = s_1$, $\tau_{k+1}^{ca} = s_2$, $\tau_{k-\tau_k^{ca}}^{sc} = r_1$ and $\tau_{k+1-\tau_{k+1}^{ca}}^{sc} = r_1$

Then, the evaluation of the terms ΔV_s, $s = 1, 2, \ldots, 5$, involves the following probability transition matrices: $[\mathrm{T}^{sc}] = \{\lambda_{mn}\}: \tau_k^{sc} \to \tau_{k+1}^{sc}$, $[\mathrm{T}^{ca}]^{m-n+1}: \tau_{k-\tau_k^{sc}-1}^{ca} \to \tau_{k-\tau_{k+1}^{sc}}^{ca}$, $[\mathrm{T}^{ca}]^{n}: \tau_{k-\tau_{k+1}^{sc}}^{ca} \to \tau_k^{ca}$, $[\mathrm{T}^{ca}] = \{\rho_{ij}\}: \tau_k^{ca} \to \tau_{k+1}^{ca}$ and $[\mathrm{T}^{sc}]^{s_1-s_2+1}: \tau_{k-\tau_k^{ca}}^{sc} \to \tau_{k+1-\tau_{k+1}^{ca}}^{sc}$.

These transition matrices involve multi-step jump of Markov chains modeling the induced delays. Then, the terms in (11) are evaluated as:

$$\begin{aligned} \Delta V_1 \triangleq\ & \mathbb{E}\left[V_1\left(x(k+1), \tau_{k+1}^{sc}, \tau_{k-\tau_{k+1}^{sc}}^{ca}, \tau_{k+1}^{ca}, \tau_{k+1-\tau_{k+1}^{ca}}^{sc} \mid x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right) - V_1(\cdot)\right] \\ &= x^T(k)\left(A^T\bar{P}(m,i)A - P(m,i)\right)x(k) + 2x^T(k)A^T\bar{P}(m,i)BK(m,i)x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right) \\ &+ x^T\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)K^T(m,i)B^T\bar{P}(m,i)B \times K(m,i)x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right) \end{aligned} \tag{12}$$

$$\begin{aligned} \Delta V_2 \triangleq\ & \mathbb{E}\left[V_2\left(x(k+1), \tau_{k+1}^{sc}, \tau_{k-\tau_{k+1}^{sc}}^{ca}, \tau_{k+1}^{ca}, \tau_{k+1-\tau_{k+1}^{ca}}^{sc} \mid x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right) - V_2(\cdot)\right] \\ &= \sum_{s_2=0}^{\tau_{max}^{ca}} \sum_{r_2=0}^{\tau_{max}^{pc}} [T^{ca}]_{s_1,s_2} \cdot [T^{sc}]_{r_1,r_2}^{r_1-r_2+1} \left(\sum_{l=k+1-s_2-r_2}^{k} x^T(l)Q_1x(l)\right) - \sum_{l=k-s_1-r_1}^{k-1} x^T(l)Q_1x(l) \end{aligned}$$

This yields to:

$$\Delta V_2 \le x^T(k)Q_1x(k) - x^T\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)Q_1x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right) + \sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}+1}^{k-\tau_{min}^{ca}-\tau_{min}^{sc}} x^T(l)Q_1x(l) \tag{13}$$

$$\begin{aligned}
&\Delta V_3 \triangleq \\
&\mathbb{E}\Big[V_3\Big(x(k+1),\tau_{k+1}^{sc},\tau_{k-\tau_{k+1}^{sc}}^{ca},\tau_{k+1}^{ca},\tau_{k+1-\tau_{k+1}^{ca}}^{sc}\,|\\
&x(k),\tau_k^{sc},\tau_{k-\tau_k^{sc}-1}^{ca},\tau_k^{ca},\tau_{k-\tau_k^{ca}}^{sc}\Big)-V_3(\cdot)\Big]\\
&=\sum_{\theta=-\tau_{max}^{ca}-\tau_{max}^{sc}+1}^{-\tau_{min}^{ca}-\tau_{min}^{sc}}\left(\sum_{l=k+\theta+1}^{k}-\sum_{l=k+\theta}^{k-1}.\right)\\
&\times x^T(l)Qx(l)_1\\
&=\begin{pmatrix}\tau_{max}^{ca}+\tau_{max}^{sc}+\\-\tau_{min}^{ca}-\tau_{min}^{sc}\end{pmatrix}x^T(k)Q_1x(k)\\
&-\sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}+1}^{k-\tau_{min}^{ca}-\tau_{min}^{sc}}x^T(l)Q_1x(l)
\end{aligned}\tag{14}$$

$$\begin{aligned}
&\Delta V_4 \triangleq \\
&\Big[V_4\Big(x(k+1),\tau_{k+1}^{sc},\tau_{k-\tau_{k+1}^{sc}}^{ca},\tau_{k+1}^{ca},\tau_{k+1-\tau_{k+1}^{ca}}^{sc}\,|\\
&x(k),\tau_k^{sc},\tau_{k-\tau_k^{sc}-1}^{ca},\tau_k^{ca},\tau_{k-\tau_k^{ca}}^{sc}\Big)-V_4(\cdot)\Big]=\\
&\left(\sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}+1}^{k}-\sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}}^{k-1}.\right)\\
&\times x^T(l)Q_2x(l)\\
&=x^T(k)Q_2x(k)+\\
&-x^T(k-\tau_{max}^{ca}-\tau_{max}^{sc})\\
&Q_2x(k-\tau_{max}^{ca}-\tau_{max}^{sc})
\end{aligned}\tag{15}$$

$$\begin{aligned}
&\Delta V_5 \triangleq \\
&\mathbb{E}\Big[V_5\Big(x(k+1),\tau_{k+1}^{sc},\tau_{k-\tau_{k+1}^{sc}}^{ca},\tau_{k+1}^{ca},\tau_{k+1-\tau_{k+1}^{ca}}^{sc}\\
&x(k),\tau_k^{sc},\ \tau_{k-\tau_k^{sc}-1}^{ca},\tau_k^{ca},\tau_{k-\tau_k^{ca}}^{sc}\Big)-V_5(\cdot)\Big]\\
&=\sum_{\theta=-\tau_{max}^{ca}-\tau_{max}^{sc}}^{-1}\left(\sum_{l=k+\theta+1}^{k}-\sum_{l=k+\theta}^{k-1}.\right)\delta^T(l)\\
&\times(Z_1+Z_2)\delta\\
&=\sum_{\theta=-\tau_{max}^{ca}-\tau_{max}^{sc}}^{-1}[\delta^T(k)(Z_1+Z_2)\delta(k)-\\
&\delta^T(k+\theta)\cdot\\
&(Z_1+Z_2)\delta(k+\theta)]\\
&=(\tau_{max}^{sc}+\tau_{max}^{ca})\delta^T(k)(Z_1+Z_2)\delta(k)\\
&-\sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}}^{k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}-1}\delta^T(l)Z_1\delta(l)\\
&-\sum_{l=k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}}^{k-1}\delta^T(l)Z_1\delta(l)\\
&-\sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}}^{k-1}\delta^T(l)Z_2\delta(l)
\end{aligned}\tag{16}$$

Let:

$$\zeta(k)=\begin{bmatrix}x(k)\\x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)\\x(k-\tau_{max}^{sc}-\tau_{max}^{ca})\end{bmatrix}.$$

Then, it is:

$$\begin{aligned}
&\Delta V=\textstyle\sum_{s=1}^{5}\Delta V_s\ \le\ x^T(k)\cdot\\
&\left(A^T\bar{P}(m,i)A-P(m,i)\right)x(k)\\
&+2x^T(k)A^T\bar{P}(m,i)BK(m,i)x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)\\
&+\ x^T\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)K^T(m,i)B^T\bar{P}(m,i)BK(m,i)\\
&\cdot x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)+(\tau_r+1)x^T(k)Q_1x(k)\\
&+x^T(k)Q_2x(k)+\\
&-x^T(k-\tau_{max}^{ca}-\tau_{max}^{sc})Q_2x(k-\tau_{max}^{ca}-\tau_{max}^{sc})\\
&+\tau_M\begin{bmatrix}(A-I)x(k)+\\+BK(m,i)x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)\end{bmatrix}^T\\
&\times(Z_1+Z_2)\begin{bmatrix}(A-I)x(k)+\\+BK(m,i)x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)\end{bmatrix}\\
&-\textstyle\sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}}^{k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}-1}\delta^T(l)Z_1\delta(l)\\
&-\textstyle\sum_{l=k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}}^{k-1}\delta^T(l)Z_1\delta(l)\\
&-\textstyle\sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}}^{k-1}\delta^T(l)Z_2\delta(l)\\
&+2\zeta^T(k)\mathcal{U}(m,i)\begin{pmatrix}x(k)-x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)+\\-\sum_{l=k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}}^{k-1}\delta(l)\end{pmatrix}\\
&+2\zeta^T(k)\mathcal{V}(m,i)\begin{pmatrix}x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)+\\-x(k-\tau_{max}^{ca}-\tau_{max}^{sc})\\-\sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}}^{k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}-1}\delta(l)\end{pmatrix}\\
&+2\zeta^T(k)\mathcal{V}(m,i)\begin{pmatrix}x\left(k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}\right)\\-x(k-\tau_{max}^{ca}-\tau_{max}^{sc})+\\-\sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}}^{k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}-1}\delta(l)\end{pmatrix}
\end{aligned}$$

It can be noted that the last three terms introduced in the right hand side of the inequality are identically nil with:

$$\mathcal{U}(m,i)\triangleq[\mathcal{U}_1^T(m,i)\ \ \mathcal{U}_2^T(m,i)\ \ \mathcal{U}_3^T(m,i)]^T,$$

$$\mathcal{V}(m,i)\triangleq[\mathcal{V}_1^T(m,i)\ \ \mathcal{V}_2^T(m,i)\ \ \mathcal{V}_3^T(m,i)]^T$$

and:

$$\mathcal{W}(m,i)\triangleq[\mathcal{W}_1^T(m,i)\ \ \mathcal{W}_2^T(m,i)\ \ \mathcal{W}_3^T(m,i)]^T.$$

After some matrix manipulations, the following equation is obtained:

$$\Delta V \leq \zeta^T(k) \times \left\{ \begin{aligned} &\begin{bmatrix} A^T\bar{P}(m,i) \\ K^T(m,i)B^T\bar{P}(m,i) \\ 0 \end{bmatrix} \bar{P}(m,i)^{-1} \begin{bmatrix} A^T\bar{P}(m,i) \\ K^T(m,i)B^T\bar{P}(m,i) \\ 0 \end{bmatrix}^T \\ &+ \begin{bmatrix} \sqrt{\tau_M}(A-I)^T Z_2 \\ \sqrt{\tau_M}K^T(m,i)B^T Z_2 \\ 0 \end{bmatrix} Z_2^{-1} \begin{bmatrix} \sqrt{\tau_M}(A-I)^T Z_2 \\ \sqrt{\tau_M}K^T(m,i)B^T Z_2 \\ 0 \end{bmatrix}^T \\ &+ \begin{bmatrix} \sqrt{\tau_M}(A-I)^T Z_1 \\ \sqrt{\tau_M}K^T(m,i)B^T Z_1 \\ 0 \end{bmatrix} \times \\ &\times Z_1^{-1} \begin{bmatrix} \sqrt{\tau_M}(A-I)^T Z_1 \\ \sqrt{\tau_M}K^T(m,i\,)B^T Z_1 \\ 0 \end{bmatrix}^T \end{aligned} \right. + \begin{bmatrix} \Lambda_{11}(m,i) & \Lambda_{12}(m,i) & \Lambda_{13}(m,i) \\ * & \Lambda_{22}(m,i) & \Lambda_{23}(m,i) \\ * & * & \Lambda_{33}(m,i) \end{bmatrix} + \left. \begin{aligned} &+\tau_M \mathcal{U}(m,i) Z_1^{-1} \mathcal{U}^T(m,i) + \\ &\tau_r \mathcal{V}(m,i) Z_1^{-1} \mathcal{V}^T(m,i) \\ &+\tau_M \mathcal{W}(m,i) Z_2^{-1} \mathcal{W}^T(m,i) \end{aligned} \right\} \zeta(k)$$
$$- \sum_{l=k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}}^{k-1} \begin{bmatrix} \zeta^T(k)\mathcal{U} + \\ +\delta^T(l) Z_1 \end{bmatrix} \times \begin{bmatrix} \zeta^T(k)\mathcal{U}(m,i) + \\ +\delta^T(l) Z_1 \end{bmatrix}^T$$
$$- \sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}}^{k-\tau_k^{ca}-\tau_{k-\tau_k^{ca}}^{sc}-1} \left(\begin{bmatrix} \zeta^T(k)\mathcal{V}(m,i) + \\ +\delta^T(l) Z_1 \end{bmatrix} Z_1^{-1} \begin{bmatrix} \zeta^T(k)\mathcal{V}(m,i) + \\ +\delta^T(l) Z_1 \end{bmatrix}^T \right)$$
$$- \sum_{l=k-\tau_{max}^{ca}-\tau_{max}^{sc}}^{k-1} \left(\begin{bmatrix} \zeta^T(k)\mathcal{W}(m,i) + \\ +\delta^T(l) Z_2 \end{bmatrix} Z_2^{-1} \begin{bmatrix} \zeta^T(k)\mathcal{W}(m,i) + \\ +\delta^T(l) Z_2 \end{bmatrix}^T \right) \tag{17}$$

Since both $Z_1 > 0$ and $Z_2 > 0$, it is concluded that the last three terms in (17) are non-positive and this leads to (18).

By applying Schur complement, equation (10) is equivalent to $\Xi(m,i)$ and guarantees $\Xi(m,i) < 0$.

Then, it is:

$$\Delta V \leq \zeta^T(k) \times \left\{ \begin{aligned} &\begin{bmatrix} A^T\bar{P}(m,i) \\ K^T(m,i)B^T\bar{P}(m,i) \\ 0 \end{bmatrix} \bar{P}(m,i)^{-1} \begin{bmatrix} A^T\bar{P}(m,i) \\ K^T(m,i)B^T\bar{P}(m,i) \\ 0 \end{bmatrix}^T \\ &+ \begin{bmatrix} \sqrt{\tau_M}(A-I)^T Z_2 \\ \sqrt{\tau_M}K^T(m,i)B^T Z_2 \\ 0 \end{bmatrix} Z_2^{-1} \begin{bmatrix} \sqrt{\tau_M}(A-I)^T Z_2 \\ \sqrt{\tau_M}K^T(m,i)B^T Z_2 \\ 0 \end{bmatrix}^T \\ &+ \begin{bmatrix} \sqrt{\tau_M}(A-I)^T Z_1 \\ \sqrt{\tau_M}K^T(m,i)B^T Z_1 \\ 0 \end{bmatrix} \times \\ &Z_1^{-1} \begin{bmatrix} \sqrt{\tau_M}(A-I)^T Z_1 \\ \sqrt{\tau_M}K^T(m,i\,)B^T Z_1 \\ 0 \end{bmatrix}^T \end{aligned} \right. + \begin{bmatrix} \Lambda_{11}(m,i) & \Lambda_{12}(m,i) & \Lambda_{13}(m,i) \\ * & \Lambda_{22}(m,i) & \Lambda_{23}(m,i) \\ * & * & \Lambda_{33}(m,i) \end{bmatrix} + \left. \begin{aligned} &+\tau_M \mathcal{U}(m,i) Z_1^{-1} \mathcal{U}^T(m,i) + \\ &\tau_r \mathcal{V}(m,i) Z_1^{-1} \mathcal{V}^T(m,i) \\ &+\tau_M \mathcal{W}(m,i) Z_2^{-1} \mathcal{W}^T(m,i) \end{aligned} \right\} \zeta(k) = \zeta^T(k)\Xi(m,i)\zeta(k) \tag{18}$$

$$\begin{aligned} \Delta V = \\ &\mathbb{E}\Big[V\left(x(k+1), \tau_{k+1}^{sc}, \tau_{k-\tau_{k+1}^{sc}}^{ca}, \tau_{k+1}^{ca}, \tau_{k+1-\tau_{k+1}^{ca}}^{sc}\right) \\ &-V\left(x(k), \tau_k^{sc}, \tau_{k-\tau_k^{sc}-1}^{ca}, \tau_k^{ca}, \tau_{k-\tau_k^{ca}}^{sc}\right)\Big] \\ &\leq -\lambda_{min}\big(-\Xi(m,i)\big)\zeta^T(k)\zeta(k) \\ &\leq -\sigma x^T(k)x(k) \end{aligned} \tag{19}$$

where $\lambda_{min}\big(-\Xi(m,i)\big)$ denotes the minimal eigenvalue of $-\Xi(m,i)$ and $\sigma = \inf\{\lambda_{min}\big(-\Xi(m,i)\big), m \in \mathcal{M}, i \in \mathcal{N}\}$. By iterating the inequality relationship in (19), it follows that for any $T > 0$:

$$\begin{aligned} &\mathbb{E}\Big[V\left(x(T+1), \tau_{T+1}^{sc}, \tau_{T-\tau_{T+1}^{sc}}^{ca}, \tau_{T+1}^{ca}, \tau_{T+1-\tau_{T+1}^{ca}}^{sc}\right)\Big] \\ &-\mathbb{E}[V(\varphi_i, -\tau_{max}^{sc} - \tau_{max}^{ca} \leq i \leq \\ &0, \tau_0^{ca}, \tau_0^{sc}, \tau_{-\tau_0^{ca}}^{sc}, \tau_{-\tau_0^{sc}-1}^{ca}\big)\big] \\ &\leq -\sigma \textstyle\sum_{k=0}^{T} \mathbb{E}[x^T(k)x(k)] \\ &-\mathbb{E}\Big[V\left(x(k+1), \tau_{T+1}^{sc}, \tau_{T-\tau_{T+1}^{sc}}^{ca}, \tau_{T+1}^{ca}, \tau_{T+1-\tau_{T+1}^{ca}}^{sc}\right)\Big] \end{aligned}$$

With $\xi = \frac{1}{\sigma}$, this inequality yields:

$$\sum_{k=0}^{T} \mathbb{E}[x^T(k)x(k)]$$
$$\leq \xi\left(\mathbb{E}\left[V\left(x(0), \tau_0^{ca}, \tau_0^{sc}, \tau_{-\tau_0^{ca}}^{sc}, \tau_{-\tau_0^{sc}-1}^{ca}\right)\right]\right.$$
$$\left.-\mathbb{E}\left[V\left(x(k+1), \tau_{T+1}^{sc}, \tau_{T-\tau_{T+1}^{sc}}^{ca}, \tau_{T+1}^{ca}, \tau_{T+1-\tau_{T+1}^{ca}}^{sc}\right)\right]\right)$$
$$\leq \xi\mathbb{E}\left[V\left(\begin{matrix}\varphi_i, -\tau_{max}^{sc} + \\ -\tau_{max}^{ca} \leq i \leq 0, \tau_0^{ca}, \tau_0^{sc}, \tau_{-\tau_0^{ca}}^{sc}, \tau_{-\tau_0^{sc}-1}^{ca}\end{matrix}\right)\right]$$

Implying for $T \to \infty$:

$$\sum^{\infty}{}_{k=0} \mathbb{E}\left[x^T(k)x(k)|x(0), \tau_0^{ca}, \tau_0^{sc}, \tau_{-\tau_0^{ca}}^{sc}, \tau_{-\tau_0^{sc}-1}^{ca}\right] \leq$$
$$\xi\mathbb{E}\left[V\left(\begin{matrix}\varphi_i, -\tau_{max}^{sc} + \\ -\tau_{max}^{ca} \leq i \leq 0, \tau_0^{ca}, \tau_0^{sc}, \tau_{-\tau_0^{ca}}^{sc}, \tau_{-\tau_0^{sc}-1}^{ca}\end{matrix}\right)\right]$$

This implies stochastic stability of the closed-loop system (7) and thus completes the proof of Theorem 1.

Theorem 1 gives sufficient LMI conditions for the closed-loop system (7) to be stochastically stable under state feedback controller (4).

However, these conditions are nonlinear in the controller gain matrices $K(m,i)$, $m \in \mathcal{M} = \{0,1, \ldots, \tau_{max}^{sc}\}$and $i \in \mathcal{N} = \{0,1, \ldots, \tau_{max}^{ca}\}$. In order to design the state feedback stabilizing controller, equivalent LMI conditions with some matrix inverse constraints are derived and given in the following theorem.

Theorem 2: For the closed-loop networked control system (7) with random but bounded network induced delays $\tau_k^{sc} \in \mathcal{M}$ and $\tau_k^{ca} \in \mathcal{N}$, there exists a controller (4) such that the system is stochastically stable, if for each mode $m \in \mathcal{M} = \{0,1, \ldots, \tau_{max}^{sc}\}$and $i \in \mathcal{N} = \{0,1, \ldots, \tau_{max}^{ca}\}$, there exist matrices $P(m,i) > 0$, $\mathcal{X}(m,i)$, $Q_v > 0$, $Z_v > 0$, $S_v > 0 v = 1,2$, $U_r(m,i)$,$V_r(m,i)$,$W_r(m,i)$, $r = 1,2,3$ and $K(m,i)$, such that:

$$\begin{bmatrix} -x & 0 & 0 & \Omega_1(m,i) \\ * & -S_2 & 0 & \Omega_2(m,i) \\ * & * & -S_2 & \Omega_2(m,i) \\ * & * & * & \Omega_3(m,i) \end{bmatrix} < 0 \tag{20}$$

$$P(m,i)\mathcal{X}(m,i) = I, Z_1S_1 = I, Z_2S_2 = I \tag{21}$$

where:

$$\mathcal{X} = diag\{\mathcal{X}_0, \mathcal{X}_1, \ . \ . \ . \ , \mathcal{X}_{\tau_{max}^{sc}}\}$$
$$\mathcal{X}_h = diag\{\mathcal{X}(h,0), \ \mathcal{X}(h,0), . \ . \ . , \mathcal{X}(h, \tau_{max}^{ca})\}$$

$$\Omega_1(m,i) =$$
$$[\mathcal{L}(m,i)A \quad \mathcal{L}(m,i)BK(m,i) \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\Omega_2(m,i =)$$
$$[\sqrt{\tau_M}(A-I) \quad \sqrt{\tau_M}BK(m,i) \quad 0 \quad 0 \quad 0 \quad 0]$$

$$\Omega_3(m,i) = \Psi_4(m,i)$$

with:

$$\mathcal{L}(m,i) = \left[\mathcal{L}_0^T(m,i) \ \mathcal{L}_1^T(m,i) \ . \ . \ . \ \mathcal{L}_{\tau_{max}^{sc}}^T(m,i)\right]^T$$
$$\mathcal{L}_h(m,i) = \left[\mathcal{L}_{h,0}(m,i)\mathcal{L}_{h,1}(m,i) \ . \ . \ . \ \mathcal{L}_{h,\tau_{max}^{ca}}(m,i)\right]^T$$

and:

$$\mathcal{L}_{h,l}(m,i) = \left(\sum_{s_1=0}^{\tau_{max}^{ca}} [T^{sc}]_{m,h}[T^{ca}]_{i,l}^{m-h+1}[T^{ca}]_{l,s_1}^{h}\right)^{\frac{1}{2}}$$

for $h = 0,1, . \ . \ ., \tau_{max}^{sc}$ and , $l = 0,1,2, . \ . \ , \tau_{max}^{ca}$.

Moreover, if (20) and (21) have solutions, the mode-dependent controller gain is given by $K(m,i)$.

Proof: By Schur complement, (10) is equivalent to:

$$\Phi_1(m,i) + \Phi_2(m,i) < 0 \tag{22}$$

$$\Phi_1(m,i) = \begin{bmatrix} -Z_2^{-1} & 0 & \Omega_2(m,i) \\ * & -Z_1^{-1} & \Omega_2(m,i) \\ * & * & \Psi_4(m,i) \end{bmatrix}$$

$$\Phi_2(m,i) = \Psi_5^T(m,i)\bar{P}(m,i)\Psi_5(m,i)$$

with:

$$\Psi_5(m,i) = [0 \quad 0 \quad A \quad BK(m,i) \quad 0 \quad 0 \quad 0 \quad 0].$$

By expressing the matrix $\bar{P}(m,i)$ introduced in theorem (1) in a compact form as follows:

$$\begin{aligned} &\bar{P}(m,i) \triangleq \\ &\sum_{n=0}^{\tau_{max}^{sc}} \sum_{j=0}^{\tau_{max}^{ca}} \sum_{s_1=0}^{\tau_{max}^{ca}} [T^{sc}]_{m,n}[\mathrm{T}^{ca}]_{i,j}^{m-n+1} [\mathrm{T}^{ca}]_{j,s_1}^{n} \\ &\times P(n,j) \\ &= \sum_{h=0}^{\tau_{max}^{sc}} \mathcal{L}_h{}^T(m,i)\hat{P}_h\mathcal{L}_h(m,i) \\ &= \mathcal{L}^T(m,i)\hat{P}\mathcal{L}(m,i) \end{aligned} \tag{23}$$

where $\hat{P}_h = diag\{P(h,0) , \ P(h,1) , . \ . \ . , P(h, \tau_{max}^{ca})\}$ and, $\hat{P} = diag\{\hat{P}_0, \hat{P}_1 , \ . \ . \ . \ , \hat{P}_{\tau_{max}^{sc}}\}$. Then, the matrix $\Phi_2(m,i)$ can be written:

$$\begin{aligned} &\Phi_2(m,i) = \Psi_5^T(m,i)\mathcal{L}^T(m,i)\hat{P}\mathcal{L}(m,i)\Psi_5(m,i) \\ &= [0 \quad 0 \quad \Omega_1(m,i)]^T\hat{P}[0 \quad 0 \quad \Omega_1(m,i)] \end{aligned} \tag{24}$$

Using (24) with the constraints (21) and by Schur complement, equation (22) is equivalent to (20). This completes the proof. The conditions stated in Theorem 2 are in fact a set of LMIs with some matrix inversion constraints. Though they are non convex, there exist methods to solve them such as the cone complementary linearization (CCL) algorithm which has been demonstrated to be efficient in numerical implementation. Hence, in this paper, it is suggested to use CCL algorithm in order to calculate the controller gains $K(m,i)$ from Theorem 2.

IV. Simulation Results

In this section, an example is presented in order to illustrate the effectiveness of the proposed state feedback controller. Consider a cart-pendulum system actuated by a permanent magnet DC motor coupled to an output gear [32] shown in Fig. 2. The combination of the DC motor dynamics with the cart-pendulum subsystem results in a four-state single input state space model. The state vector is defined as $x(t) = [y(t)\dot{y}(t)\theta(t)\dot{\theta}(t)]^T$ where $y(t)$ is the position of the cart on the track and $\theta(t)$ is the angle that pendulum makes with the upright equilibrium. The input to the system is the voltage $v(t)$applied to the DC motor. The cart track surface is assumed frictionless and the system parameters are summarized in Table I [32].

The state feedback controller is designed for the following linearized discrete-time model obtained with a sampling time $T_s = 0.025$s:

$$A = \begin{bmatrix} 1.0000 & 0.0208 & -0.0011 & 0 \\ 0 & 0.6808 & -0.0817 & -0.0011 \\ 0 & 0.0085 & 1.0083 & 0.0251 \\ 0 & 0.6399 & 0.6553 & 1.0083 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0009 \\ 0.0714 \\ -0.0019 \\ -0.1432 \end{bmatrix}$$

The eigenvalues of A are 1.0000, 1.1228, 0.6720 and 0.9026. Hence, the discretized system is unstable. The random delays in the S-C and C-A links of the NCS vary as $\tau_k^{sc} \in \{0,1\}$ and, $\tau_k^{ca} \in \{0,1,2\}$ respectively. The transition probability matrices are given by:

$$\mathrm{T}^{sc} = \begin{bmatrix} 0.20 & 0.80 \\ 0.15 & 0.85 \end{bmatrix}, \quad \mathrm{T}^{ca} = \begin{bmatrix} 0.10 & 0.70 & 0.20 \\ 0.15 & 0.55 & 0.30 \\ 0.20 & 0.40 & 0.40 \end{bmatrix}$$

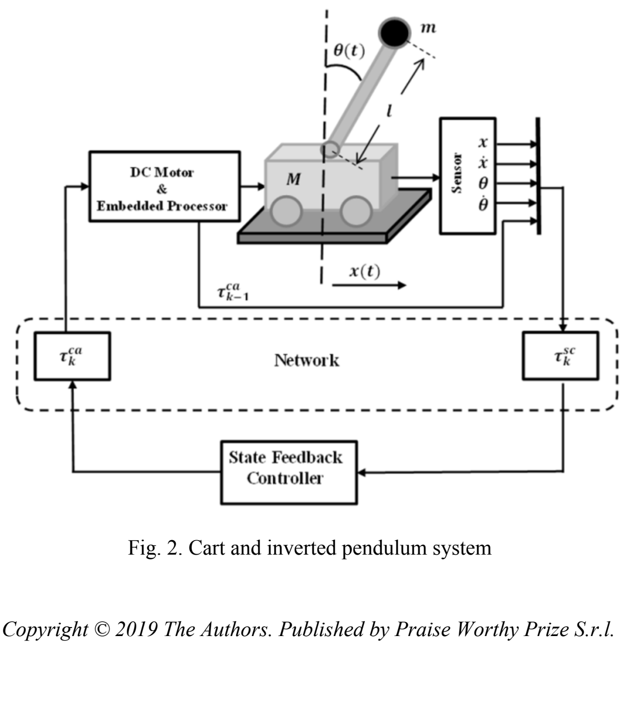

Fig. 2. Cart and inverted pendulum system

TABLE I
INVERTED PENDULUM SYSTEM PARAMETERS

Parameter	Description	Unit	Value
M	Mass of cart	kg	0.5
m	Mass of pendulum	kg	0.2
l	Distance to pendulum center of gravity	m	0.5
g	Gravitational acceleration	ms^{-2}	9.81
R_m	Motor armature resistance	Ω	2.6
K_m	Back EMF constant	V(rad s^{-1})	0.0076
K_g	Gear ratio (driven gear/ driving gear)		3.7
r	Radius of output gear	m	0.0063

Based on Theorem 2 and using the CCL algorithm, the two-mode state feedback controller gains are:

$$\begin{aligned} K(0,0) &= [0.0095 \quad 4.7943 \quad 9.5007 \quad 2.0504], \\ K(0,1) &= [0.0403 \quad 4.7707 \quad 9.8760 \quad 2.0095], \\ K(0,2) &= [0.0221 \quad 4.7893 \quad 9.6776 \quad 2.0190], \\ K(1,0) &= [0.0568 \quad 4.7582 \quad 10.1068 \quad 2.1051], \\ K(1,1) &= [0.0685 \quad 4.7547 \quad 10.3650 \quad 2.1702], \\ K(1,2) &= [0.0643 \quad 4.7561 \quad 10.2680 \quad 2.1568]. \end{aligned}$$

The state trajectories of the discrete-time model and the state feedback controller are obtained for initial state values:

$$\begin{aligned} x(-3) &= x(-2) = x(-1) \\ &= x(0) = [-0.35 \;\; 0 \;\; 0.15 \;\; 0]^T \end{aligned}$$

Initial delays are set to $\tau_0^{sc} = \tau_0^{ca} = 0$.

Figures 3 and 4 show the jumps of the S-C τ_k^{sc} and C-A τ_k^{ca} delays during the first five seconds of the simulation run according to their respective transition probabilities. The responses of the cart-pendulum actuated by a DC motor under the proposed two-mode dependent state feedback controller are illustrated by Figures 5 and 6. As shown in Figures 5 and 6, the position and the velocity stabilize in less than 5 seconds.

The startup oscillations in velocity are due to load inertia and transient phenomena of the DC motor. It can be seen that the closed loop system is stochastically stable. The transition time of the pendulum angular position is shown to be lower than the time taken by the cart position. The generated control signal applied to the DC motor is shown in Fig. 7. The peak value of the transition DC input is reached during the starting time.

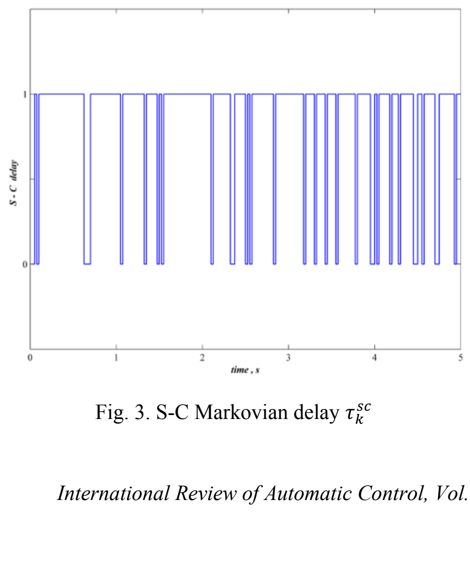

Fig. 3. S-C Markovian delay τ_k^{sc}

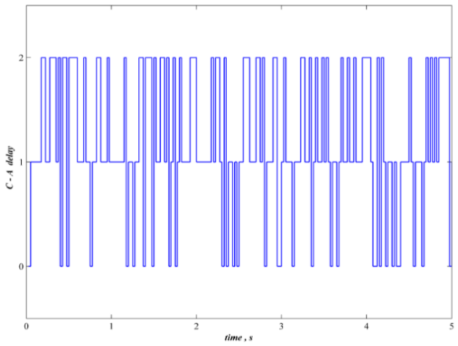

Fig. 4. C-A Markovian delay τ_k^{ca}

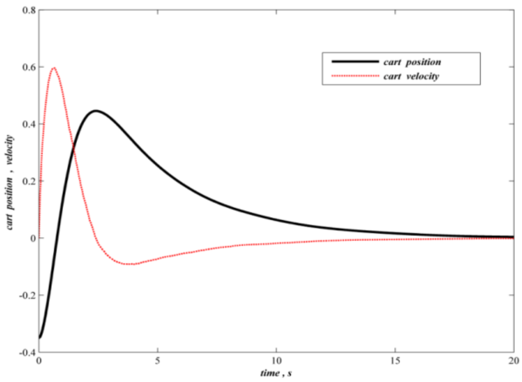

Fig. 5. Responses of cart position $y(t)$, and velocity $\dot{y}(t)$

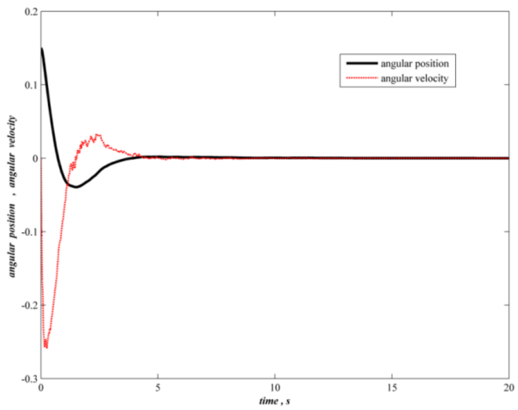

Fig. 6. Responses of pendulum angular position $\theta(t)$ and angular velocity $\dot{\theta}(t)$

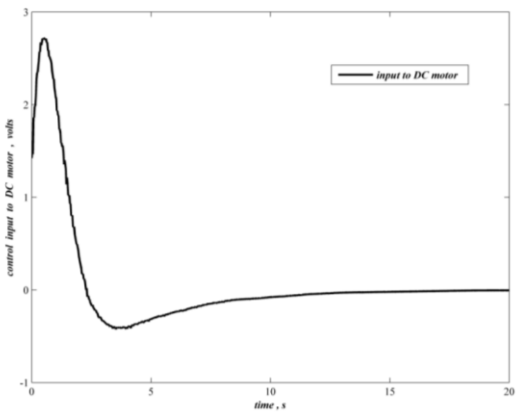

Fig. 7. Stabilizing control signal at DC motor input

V. Conclusion

The stability problem for a class of networked control systems (NCSs) with the plant is considered as a Markovian jump system. The random delays from the sensor to the controller and from the controller to the actuator are modeled as two Markov chains. In this paper, a two-mode state feedback controller has been proposed for networked control systems with S-C and C-A random communication delays within the general framework of discrete-time Markovian jump linear systems. By applying a type of stochastic Lyapunov functional, sufficient conditions on the stochastic stabilizability are derived in terms of coupled LMIs.

These conditions incorporate the full effect of the C-A delay on the closed loop system into the controller design. In order to reduce the conservativeness of the stabilization conditions of the NCS, the full effect of the C-A delay on the state signal is incorporated in the closed-loop model. Numerical Simulations are presented in order to demonstrate the effectiveness of the controller. A stabilizing control signal at DC motor input is achieved under minimum surge voltage in a short transition period. It should be pointed out that the results obtained are encouraging, and it is worth mentioning that the proposed scheme can be extended in order to consider observer-based networked control systems with random delays.

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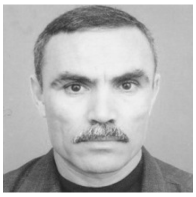

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